Weak* Topology

P. Sam Johnson

July 2, 2020

Overview

We consider the underlying space as **the dual space of a normed space X.**

We have seen that X^* has the norm and weak topologies.

We now disucss another topology of X^*.

Definition 1.

Let X be a normed space and Q be the natural map from X into X^{**}. Then the topology for X^* induced by the topologizing family $Q(X)$ is the weak (pronounced "weak star") topology of X^* or the X topology of X^* or the toplogy $\sigma(X^*, X)$.*

That is, the weak* topology of X^* is the smallest topology for X^* such that, for each x in X, the linear functional $F_x : x^* \to x^*x$ on X^* is continuous with respect to that topology. Note that
$Q(X) = \{F_x : x \in X\}$.

Notation

Whenever w^* (pronounced "weak star") is attached to a topological symbol, it indicates that the reference is to the weak* topology. For instance, $x_\alpha^* \xrightarrow{w^*} X^*$, $w^*\text{-}\lim_\alpha x_\alpha^* = x^*$, $\overline{A}^{w^*}$ and so on.

Note that $Q(X)$ separates the points of X^*, we have the following.

Theorem 2.

Let X be a normed space. Then the weak topology of X^* is a completely regular locally convex subtopology of the weak topology of X^*, and therefore of the norm topology of X^*. Moreover, the dual space of X^* with respect to this topology is $Q(X)$.*

Weak* Topology

Note that $Q(X) \subseteq X^{**}$, in general.

Theorem 3.

The weak* and weak topologies of X^* are the same if and only if X is reflexive (That is, $Q(X) = X^{**}$).

Corollary 4.

Let X be a normed space. Then the weak* and norm topologies of X^* are the same if and only if X is finite dimensional.

Theorem 5.

Let X be a normed space. Then a linear functional on X^* is weakly* continuous if and only if has the form $x^* \mapsto x^* x_0$ for some x_0 in X.

That is, weak* topology is the weakest topology on the dual X^* for which only continuous functions are the element of the space X.

Weak and weak* topologies of X^*

The weak and weak* topologies of X^* may be different. By the above theorem, the space should not be reflexive. We know that the space c_0 is not reflexive and the dual space of c_0 is ℓ_1.

Example 6.

Let (e_n) be the standard basis of c_0. Generally (e_n^*) is the notion for bi-orthogonal sequence for (e_n), that is, $e_m^*(e_n) = 1$ if $n = m$, 0 otherwise. So, in the case (e_n^*) is again the sequence whose nth entry is 1 and rest are 0. That is, (e_n^*) is the sequence of elements of c_0^* that corresponds in the usual way to the standard unit vectors of ℓ_1.

Now take any element $x = (x_n)$ of c_0 so by definition (x_n) will converge to zero. Consider the dual action of e_n^* on x, it will give x_n. So, $e_n^*(x)$ converges to zero for any x in c_0. This will give (e_n^*) is weakly convergent to 0. As the natural isometric isomorphism from ℓ_1 onto c_0^* is also a weak-to-weak homeomorphism, (e_n^*) does not converge weakly to 0. Therefore the weak and weak* topologies of c_0^* are different.*

Weakly* Convergent / Cauchy Nets

As with the weak topology of a normed space, the results obtained for a Hausdorff locally convex topology induced by a separating vector space of linear functionals all hold for the weak* topology of the dual space X^* of a normed space X.

Let (x_α^*) be a net in X^*.

1. If x^* is a member of X^*, then (x_α^*) is weakly* convergent to x^* iff $x_\alpha^* x \to x^* x$ for each x in X.
2. (x_α^*) is weakly* Cauchy iff $(x_\alpha^* x)$ is weakly* Cauchy (that is, convergent) net in $\mathbb{F}$ for each x in X.

A Basis for Weakly* Topology

A basis for the weakly* topology of X^* is given by the collection of all subsets of X^* of the form

$$\left\{ y^* \in X^* : |F_x(y^* - x^*)| = |(y^* - x^*)(x)| < 1 \text{ for each } \ x \text{ in } A \right\}$$

such that $x^* \in X^*$ and A is a finite subset of X.

Annihilators

Let X be a normed space and let A and B be subsets of X and X^* respectively.

Define $A^\perp$ and ${}^\perp B$ (pronounced "A perp" and "perp B") by the formulas

$$A^\perp = \{x^* : x^* \in X^*, x^*x = 0 \text{ for each } x \text{ in } A\}$$

$${}^\perp B = \{x : x \in X, x^*x = 0 \text{ for each } x^* \text{ in } B\}.$$

Then $A^\perp$ **is the annihilator of** A **in** X^* ;

while ${}^\perp B$ **is the annihilator of** B **in** X.

Annihilators

Since the dual X^* of a normed space X is itself a normed space, both $\perp B$ and $B \perp$ are defined for each subset B of X^*, and both have the right to be called the annihilator of B.

Notation : The space in which the annihilator is being taken should be made explicit, either by adding the qualifying phrase or by using the left-hand or right-hand "perp" notation.

Proposition 7.

Let X be a normed space and let A and B be subsets of X and X^* respectively.

1. *The sets $A^\perp$ and ${}^\perp B$ are closed subspaces of X^* and X respectively.*
2. ${}^\perp(A^\perp) = [A]$, *the smallest closed set containing A.*
3. *If A is a subspace of X, then ${}^\perp(A^\perp) = \overline{A}$.*

Exercise 8.

Let X be a normed space and let B be a subset of X^*.

1. *Prove that* ${}^{\perp}(B^{\perp}) \subseteq ({}^{\perp}B)^{\perp}$.

   For the rest of this exercise, let $X = c_0$ *and let* B *be the subset of* X^* *that corresponds to the set* $\{(\alpha_n) : (\alpha_n) \in \ell_1, \sum_n \alpha_n = 0\}$ *when* c_0^* *and* ℓ_1 *are identified in the usual way.*
2. *Show that* ${}^{\perp}(B^{\perp}) = B$.
3. *Show that* $({}^{\perp}B)^{\perp} = X^*$. *Thus, the inclusion in 1 may be proper, even when* B *is a closed subspace of* X^*.

Since the weak* topology of the dual space of a normed space X is a subtopology of the norm topology of X^*, part (a) of the following result is a strengthening of part (a) of the proposition 7.

Theorem 9.

Let X be a normed space and let A and B be subsets of X and X^* respectively.

1. *The set $A^\perp$ is a weakly* closed subspace of X^*.*
2. $({}^\perp B)^\perp = [B]^{w^*}$.
3. *If B is a subspace of X^*, then $({}^\perp B)^\perp = \overline{B}^{w^*}$.*

Definition 10.

A subset A of a normed space X is **weakly* bounded** *if, for each weak* neighbourhood U of 0 in X^*, there is a positive s_U such that*

$$A \subseteq tU$$

whenever $t > s_U$.

Recall :

Theorem 11 (A useful test for boundedness with respect to $\mathcal{F}$-topology).

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Then a subset A of X is bounded with respect to the X'-topololgy if and only if $f(A)$ is bounded in $\mathbb{F}$ for each f in X'.

Weak* bounded

Proposition 12.

Let A be a subset of a normed space X. The set A is **weakly* bounded** *if and only if $\{x^*x : x^* \in A\}$ is bounded in $\mathbb{F}$ for each x in X.*

The following result is the weak* analog of the fact that a subset of a normed space is bounded iff it is weakly bounded.

However, notice the requirement that X be a Banach space.

Theorem 13.

Let X be a **Banach space**. *Then a subset of X^* is bounded if and only if it weakly* bounded.*

Corollary 14.

Let X be a **Banach space**. *Then a subset A of X^* is bounded if and only if $\{x^*x : x^* \in A\}$ is a bounded set of scalars for each x in X.*

Recall : Vector Topology

Theorem 15.

Every compact subset of a topological vector space is bounded. Thus, every convergent sequence in a topological vector space is bounded.

Theorem 16.

Every Cauchy sequence in a topological vector space is bounded.

Theorem 17.

Every convergent net in a topological vector space is Cauchy.

Corollary 18.

Let X be a **Banach space***. Then weakly* compact subsets of X^* are weakly* bounded (or, simply bounded).*

Corollary 19.

Let X be a **Banach space***. Then weakly* Cauchy sequences in X^* is weakly* bounded (or, simply bounded).*

Corollary 20.

Let X be a **Banach space***. Then weakly* convergent sequences in X^* is weakly* bounded (or, simply bounded).*

Neither the preceding theorem nor any of its four corollaries remain true if X is only required to be a normed space.

Example 21.

Let X be the vector space of finitely nonzero sequences equipped with the ℓ_1 norm. For each positive integer m, let $x_m^* : X \to \mathbb{F}$ be defined by the formula $x_m^*((\alpha_n)) = m.\alpha_m$. Let $A = \{x_m^* : m \in \mathbb{N}\}$.

1. *Show that A is a weakly* bounded subset of X^* that is not norm bounded, and therefore that the conclusions of Theorem (13) and Corollary (14) do not follow when X is only required to be a normed space.*
2. *Show that the conclusions of Corollaries (18), (19) and (20) do not follow when X is only required to be a normed space.*

Every nonempty weakly* open subset of the dual X^* of an infinite-dimensional normed space is unbounded.

The following result does hold for every normed space, whether or not it is complete.

We proved that every nonempty weakly open subset of an infinite-dimensional normed space is unbounded. In a similar way every nonempty weakly* open subset of an infinitely-dimensional normed space is unbounded.

Theorem 22.

Let X be an infinite-dimensional normed space. Then every nonempty weakly* open subset of X^* is unbounded.

Every weakly* open subset of X^* is too big.

Recall

We discussed the following two results eariler.

Theorem 23.

The weak topology of a normed space is induced by a metric if and only if the space is finite dimensional.

Theorem 24.

The weak topology of a normed space is complete if and only if the space is finite dimensional.

Weak* analogs of the above two results

We now discuss the weak* analogs of the above two results.

Theorem 25.

Let X be a **Banach space**. *Then the weak* topology of X^* is induced by a metric if and only if X is finite dimensional.*

A completeness hypothesis in the preceding result cannot be omitted.

Exercise 26.

Give an example of an infinite-dimensional normed space X such that the weak* topology of X^* is metrizable.

Theorem 27.

Let X be a normed space. Then the weak* topology of X^* is complete if and only if X is finite dimensional.

Norm Functions

Just as norm functions are weakly lower semicontinuous, norm functions on dual spaces are weakly* lower semicontinuous.

Theorem 28.

Let X be a normed space. If (x_α^*) is a weakly* convergent net in X^*, then $\|w^*\text{-}\lim_\alpha x_\alpha^*\| \leq \liminf_\alpha \|x_\alpha^*\|$.

The following result is the partial converse of the preceding theorem.

Theorem 29.

Suppose that X is a normed space, that $\|.\|_a$ is a norm on X^* equivalent to its usual dual norm, and that $\|w^*\text{-}\lim_\alpha x_a^*\| \leq \liminf_\alpha \|x_\alpha^*\|_a$ whenever (x_α^*) is a weakly* convergent net in X^*. Then there is a norm $\|.\|_b$ on X equivalent to its original norm such that $\|.\|_a$ is the dual norm on $(X, \|.\|_b)$.

Some fundamental ways in which the two topologies differ

So far, most of the results have emphasized the similarities between the weak and weak* topologies, especially when the weak* topology is for the dual space of a Banach space.

We discussed the following result. The weak* analog of the result does not hold.

Theorem 30.

The closure and weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.

Some fundamental ways in which the two topologies differ

As the following two examples show, there are also some fundamental ways in which the two topologies differ.

Example 31.

Let X be a nonreflexive Banach space and let x^{**} be any member of x^{**} be any member of X^{**} that is not in the image of X under the natural map from X into X^{**}.
Since x^{**} is continuous but not weakly continuous, the kernel of x^{**} is a closed convex subset of X^* that is not weakly* closed. Thus, the weak* analog of (30) does not hold.*

Theorem 32 (The Banach-Alaoglu Theorem).

[S. Banach, 1932, L. Alaoglu, 1940] Let X be a normed space. Then B_{X^*} is weakly compact.*

Corollary 33.

Let X be a normed space. Then every bounded subset of X^* is relatively weakly compact. In particular, subsets of X^* that are bounded and weakly* closed are weakly* compact.*

Corollary 34.

Let X be a separable normed space and let A be a bounded subset of X^*. Then the relative weak topology of A is induced by a metric.*

Corollary 35.

Let X be a Banach space. Then every weakly* Cauchy sequence in X^* is weakly* convergent. That is, every Banach space has a weakly* sequentially complete dual space.

Corollary 36.

Let X be a normed space. Then there is a compact Hausdorff space K such that X is isometrically isomorphic to a subspace of $C(K)$. If X is a Banach space, then X is isometrically isomorphic to a closed subspace of $C(K)$.

Theorem 37.

Let X be a normed space. Then the relative weak* topology of B_{X^*} is induced by a metric if and only if X is separable.

Theorem 38.

The natural map Q from a normed space X into X^{**} is weak-to- weak* continuous, and in fact is a weak-to-relative- weak* homeomorphism from X onto $Q(X)$.

Corollary 39.

Let X be a normed space and let Q be the natural map from X into X^{**}. Then the topologies that $Q(X)$ inherits from the weak and weak* topologies of X^{**} are the same.

Theorem 40 (Goldstine's Theorem, 1938).

Let X be a normed space and Q be the natural map from X into X^{**}. Then $Q(B_X)$ is weakly* dense in $B_{X^{**}}$.

Corollary 41.

Let X be a normed space and let Q be the natural map from X into X^{**}. Then $\overline{Q(B_X)}^{w^*} = B_{X^{**}}$.

Corollary 42.

Let X be a normed space and let Q be the natural map from X into X^{**}. Then $Q(X)$ is weakly* dense in X^{**}.

References

1. Robert E. Megginson, An Introduction to Banach Space Theory, Springer, 1991.